

Singularities of bihamiltonian systems

Alexey Bolsinov and Anton Izosimov

Dept. of Math. Sciences, Loughborough University
Loughborough, LE11 3TU UK

1 Introduction

1.1 Statement of the problem

A system is called *bihamiltonian* if it is hamiltonian with respect to a whole one-dimensional family of Poisson brackets (*Poisson pencil*). A relation between bihamiltonian structure and integrability was first noted by Magri in [1] and further studied by Gelfand and Dorfman in [2], Magri and Morosi in [3], and Reiman and Semenov-Tyan-Shanskii in [4].

The principle expressed in the mentioned works is the following: given a Poisson pencil we are able to construct a large family of functions in involution which appear to be integrals of any system bihamiltonian with respect to the pencil. In many cases these integrals are enough for integrability. Therefore, bihamiltonian systems are often completely integrable.

On the other hand, many integrable systems appearing in geometry and physics possess a bihamiltonian structure (see, for example, [5–12]).

So far bihamiltonian structure was considered only as a tool for finding integrals or as an explanation for their existence, a hidden mechanism which rules integrability. But what is integrability? Integrability of a system doesn't mean that we understand its behavior - it only means a possibility to understand it. In [12] Bolsinov and Oshemkov showed that bihamiltonian structure may be very useful not only for finding integrals, but for further analysis of a system as well.

The theory of qualitative (or topological) analysis of finite-dimensional integrable hamiltonian systems is due to the works of Fomenko and his school (see [13–15]), Lerman and Umanskiy (see [16–20]), and Kharlamov (see [21]). The main idea of this theory is to replace the study of the system itself by the study of the map given by the commuting integrals of the system - *the moment map*, and of the foliation of the phase space into the connected components of the moment map level sets - *the Liouville foliation*. By Arnold-Liouville theorem¹ almost all fibers of the Liouville foliation are tori, these tori are invariant with respect to the system and the dynamics on these tori is quasi-periodic. Consequently, if one aims to understand the dynamics of an integrable hamiltonian system, it is very important to study the moment map and the topology of the Liouville foliation.

Now note that in bihamiltonian case the integrals are completely defined by the Poisson pencil. Consequently, all the properties of the moment map and the Liouville foliation are contained in the pencil itself, and since Poisson pencils usually have algebraic nature, it should be easier to study them compared to the explicit study of moment maps. The problem is to reformulate the necessary properties in the language of pencils.

¹This theorem is due to Jacobi [22], Liouville [23] and Mineur [24]. In the modern formulation it first appeared in [25]. See also [15, 26, 27].

We emphasize that while integrable systems are considered to be the most “symmetric” among all dynamical systems, the systems possessing a bihamiltonian structure are even more “symmetric”. Therefore, applying general methods of the theory of integrable systems to bihamiltonian systems seems to be unreasonable, and a separate theory should be developed. An attempt to develop such a theory is made in this paper.

Which Poisson pencils do we consider? There are two essentially different types of Poisson pencils:

- Non-degenerate (symplectic): brackets forming a pencil are non-degenerate. This situation was studied in [2, 3]. The integrals in this case are the traces of powers of the recursion operator.
- Degenerate: brackets forming a pencil are degenerate. This situation was studied in [4]². The integrals in this case are the Casimir functions of the brackets of the pencil.

Since the integrals in the non-degenerate and degenerate situation have different nature, these two cases should be treated separately. In this paper we are going to discuss the second situation. The criteria for complete integrability in this case is given by Bolsinov (see [5]). We will assume that this criteria is satisfied. This is the case, for example, for the Manakov top as well as for the systems constructed by the argument shift method on semisimple (and many other) Lie algebras.

What properties of a moment map are we interested in? The most important property of a moment map is certainly the structure of its singular set. Why are singularities so important?

- All special (and most interesting) solutions, such as equilibrium points, belong to the singular set.
- Arnold-Liouville theorem claims that all regular (i.e. those which contain no singular points of the moment map) compact fibers of the Liouville foliation are tori and the foliation in the neighborhood of a torus is trivial. Therefore, the topology of the Liouville foliation is mainly defined by its singularities.

For a general integrable system it is not easy to describe the singular set. In order to do this, we need to find those points in which the rank of the moment map Jacobi matrix drops. This involves solving algebraic equations (provided the integrals are polynomial). The number and the degree of these equations grows with the dimension and the solution procedure can become very tricky. However, if the system possesses a bihamiltonian structure, this reduces to a procedure of describing the singular points of the pencil, i.e. the union of singular points of all brackets of the pencil, which is much easier in examples than the calculations with the Jacobi matrix (Bolsinov, [5]).

Probably the second important step after finding the singular points is to check their non-degeneracy. Non-degenerate singular points are, in some sense, generic singular points. The notion of non-degenerate singular point of an integrable system is analogous to the notion of Morse singular point of a smooth function. Instead of the Morse lemma we have the Eliasson theorem here: the Liouville foliation in the neighborhood of a non-degenerate singular point is symplectomorphic to the foliation given by the quadratic parts of the integrals. The complete invariant of the Liouville foliation in the neighborhood of a non-degenerate singular point is the

²However, the construction of integrals in this case can be viewed as a generalization of the argument shift method, introduced by A.S.Mischenko and A.T.Fomenko in [28], and the Manakov construction of the integrals in the Euler case of multidimensional rigid body dynamics (see [29]).

(Williamson) *type* of the point - three non-negative integers k_e, k_h, k_f . The notion of type of a singular point of an integrable system is analogous to the notion of an index of a Morse singular point³.

Thus knowing the type of a singular point makes it possible to describe the Liouville foliation as well as dynamics in the neighborhood of a point⁴. Therefore, the first thing one should do after describing the set of singular points is to check whether these points are non-degenerate and find their type. For an arbitrary system this involves some non-trivial calculations. However, it turns out that in the bihamiltonian case the answer can be expressed in terms of the brackets of the pencil linearized at a singular point. For rank zero and corank one points the problem was approached in [30]⁵. To solve the problem in the general setting, we introduce the notion of linearization of a Poisson pencil at a singular point, which is again a Poisson pencil, but a linear one. The problem of non-degeneracy and type for an initial pencil is reduced to the same problem for the linearized one, while the linear problem can be easily solved in algebraic terms.

1.2 Integrability and non-degeneracy

Let $\dot{x} = \text{sgrad } H$ be a hamiltonian system on a symplectic manifold (M^{2n}, ω) , where

$$\text{sgrad } H = \omega^{-1} dH.$$

Let \mathcal{F} be a family of pairwise commuting integrals of the system. It will be convenient to assume that \mathcal{F} is a vector space, i.e. is closed under addition and multiplication by numbers. If it is not so, we can always replace \mathcal{F} with the space linearly spanned by \mathcal{F} .

Definition 1.

1. \mathcal{F} is said to be *complete* on M^{2n} if $\dim d\mathcal{F}(x) = n$ for almost all $x \in M^{2n}$, where $d\mathcal{F}(x) = \{df(x), f \in \mathcal{F}\}$.
2. The hamiltonian system $\dot{x} = \text{sgrad } H$ is called in this case *completely (Liouville) integrable*, or simply *integrable*⁶.
3. The foliation of M into the connected components of the common level sets $\{\mathcal{F} = \text{const}\}$ is called in this case *the Liouville foliation*.

Remark 1.1. This definition coincides with the classical one if it is possible to choose n functions in \mathcal{F} such that their differentials span $d\mathcal{F}$. Locally this can always be done.

Definition 2. A point x is called singular for a given integrable hamiltonian system if $\dim d\mathcal{F}(x) < n$. The number $\dim d\mathcal{F}(x)$ is called *the rank* of a singular point x . A fiber of the Liouville foliation, which contains at least one singular point, is called singular. All other fibers are called regular.

³See Section 1.2 for precise definitions of non-degeneracy and type.

⁴For example, knowing the type of a point, we can study its Lyapunov stability. Suppose we have a non-degenerate singular point, which is a fixed point of our system. Then, provided the system is non-resonant, this point is stable if and only if it has rank zero and it has a so-called elliptic type, which means that $k_h = k_f = 0$. The same is true for periodic trajectories, with the only difference that rank zero should be replaced with rank one.

⁵See also [31], where the results of [30] are applied to study the Zhukovskii-Volterra system.

⁶More precisely, the system $\dot{x} = \text{sgrad } H$ is called integrable if \mathcal{F} is complete and the following additional condition is satisfied: the vector field $\text{sgrad } f$ is complete for each $f \in \mathcal{F}$. However, since all our considerations are local, we omit this condition in the definition.

The Arnold-Liouville theorem claims that all compact regular fibers of a Liouville foliation are tori, and the dynamics on these tori is conditionally periodic.

Despite the fact that almost all fibers of a Liouville foliation are regular, it is also important to understand the topology and dynamics in the neighborhood of singular fibers due to the following reasons:

1. Singular fibers correspond to special regimes of motion. In particular, fixed points of a system always belong to singular fibers.
2. It is mainly singular fibers which define the global topology of a system.

Let us now recall what a non-degenerate singular point is (see [15]).

Suppose that $f \in \mathcal{F}$, $df(x) = 0$. Then we can consider the linearization of the vector field $\text{sgrad } f$ at the point x . Denote it by A_f . Since the flow defined by $\text{sgrad } f$ preserves the symplectic structure, $A_f \in \mathfrak{sp}(T_x M)$.

Now consider the space $W = \{\text{sgrad } f(x), f \in \mathcal{F}\}$. Since the functions in \mathcal{F} are in involution, all operators A_f vanish on W . Consequently, we can consider A_f as operators on W^\perp/W . Since W is isotropic, W^\perp/W carries a natural symplectic structure and $A_f \in \mathfrak{sp}(W^\perp/W)$. Also note that all A_f commute, therefore the set

$$A_{\mathcal{F}} = \{A_f, f \in \mathcal{F}, df(x) = 0\}$$

is a commutative subalgebra in $\mathfrak{sp}(W^\perp/W)$.

Definition 3. A singular point x is called *non-degenerate*, if the subalgebra $A_{\mathcal{F}}$ constructed above is a Cartan subalgebra in $\mathfrak{sp}(W^\perp/W)$.

If A is an element of a Cartan subalgebra \mathfrak{h} in \mathfrak{sp} , then its eigenvalues have the form

$$\begin{aligned} &\pm \lambda_1 i, \dots, \pm \lambda_{k_e} i, \\ &\pm \nu_1, \dots, \pm \nu_{k_h}, \\ &\pm \mu_1 \pm \xi_1 i, \dots, \pm \mu_{k_f} \pm \xi_{k_f} i. \end{aligned}$$

The triple (k_e, k_h, k_f) is the same for almost all $A \in \mathfrak{h}$. Let us call this triple a (Williamson) type of the Cartan subalgebra \mathfrak{h} . All Cartan subalgebras of the same type are conjugated to each other (Williamson, [32]).

Definition 4. The *type* of a singular point x is the type of the Cartan subalgebra $A_{\mathcal{F}} \subset \mathfrak{sp}(W^\perp/W)$ constructed above.

It is easy to see that for every non-degenerate singular point x the following equality holds: $k_e + k_h + 2k_f = n - \text{rank } x$.

Let us now state the Eliasson theorem about the linearization of Liouville foliation in the neighborhood of a non-degenerate singular point.

Definition 5.

1. The foliation which is given by the function $p^2 + q^2$ in the neighborhood of the origin in $(\mathbb{R}^2, dp \wedge dq)$ is called an elliptic (or center) singularity.
2. The foliation given by the function pq in the neighborhood of the origin in $(\mathbb{R}^2, dp \wedge dq)$ is called a hyperbolic (or saddle) singularity.

3. The foliation given by the commuting functions $p_1q_1 + p_2q_2, p_1q_2 - q_1p_2$ in the neighborhood of the origin in $(\mathbb{R}^4, dp \wedge dq)$ is called a focus-focus singularity.

Theorem 1 (Eliasson, see [33–35] for proof, see also [15]). *The Liouville foliation in the neighborhood of a non-degenerate singular point of rank r and type (k_e, k_h, k_f) is locally fiberwise symplectomorphic to the direct product of k_e elliptic, k_h hyperbolic and k_f focus-focus singularities, multiplied by a trivial foliation $\mathbb{R}^r \times \mathbb{R}^r$.*

1.3 Jordan-Kronecker theorem

It is well known that two bilinear symmetric forms, one of which is positive definite, can be simultaneously diagonalized. A similar statement holds for skew-symmetric forms:

Theorem 2 (Jordan-Kronecker theorem, see [36–38]). *Let A, B be two skew-symmetric forms on a complex vector space V . Assume that B is a generic form in the pencil $\alpha A + \beta B$, i.e.*

$$\text{rank } B \geq \text{rank } (\alpha A + \beta B) \text{ for all } \alpha, \beta.$$

Then there is a basis in V such that A, B will have the following block-diagonal form:

$$A = \begin{pmatrix} 0 & J_{k_1, \lambda_1} & & & \\ -J_{k_1, \lambda_1}^T & 0 & & & \\ & & \ddots & & \\ & & & 0 & J_{k_m, \lambda_m} \\ & & & -J_{k_m, \lambda_m}^T & 0 \\ & & & & & A_K \end{pmatrix}, \quad (1)$$

$$B = \begin{pmatrix} 0 & -E_{k_1} & & & \\ E_{k_1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & -E_{k_m} \\ & & & E_{k_m} & 0 \\ & & & & & B_K \end{pmatrix},$$

where

- $J_{k, \lambda}$ is the $k \times k$ Jordan block with the eigenvalue λ .
- E_k is the $k \times k$ identity matrix.
- $\text{rank}(A_K + \lambda B_K)$ does not depend on λ .

Remark 1.2. If B is not a generic form, one should replace it with a suitable linear combination $\alpha A + \beta B$.

Remark 1.3. The theorem also gives a normal form for A_K and B_K , which we do not need: they are block-diagonal, and the corresponding blocks are called *Kronecker blocks*.

The following property of the Jordan-Kronecker form is very important:

Proposition 1.1.

$$\text{rank}(A + \lambda B) < \max_{\nu} \text{rank}(A + \nu B) \Leftrightarrow \lambda \in \{\lambda_i\}_{1 \leq i \leq m},$$

where λ_i are the numbers entering matrix (1).

Note that we will not use the Jordan-Kronecker form in formulations of theorems, because this form can hardly be calculated explicitly in examples. Nevertheless, it will be convenient to use this form for an illustration of some notions we are going to introduce.

1.4 Bihamiltonian systems. Construction of family \mathcal{F}

Definition 6. Two Poisson brackets P_0, P_∞ (on a smooth manifold M) are called *compatible* if any linear combination of them is a Poisson bracket again. The set of non-zero linear combinations $\Pi = \{\alpha P_0 + \beta P_\infty\}$ is called in this case a *Poisson pencil*.

Remark 1.4. Sometimes it will be necessary to consider complex values of α and β . In this case $\alpha P_0 + \beta P_\infty$ must be treated as a complex-valued Poisson bracket on complex-valued functions. The corresponding Poisson tensor in this case is a two-form on the complexified cotangent space at each point.

Since it only makes sense to consider Poisson brackets up to proportionality, we will write Poisson pencils in the form

$$\Pi = \{P_\lambda = P_0 + \lambda P_\infty\}_{\lambda \in \mathbb{C}}.$$

Definition 7. Rank of a pencil Π at a point $x \in M$ is the number

$$\text{rank } \Pi(x) = \max_{\lambda} \text{rank } P_\lambda(x).$$

Rank of a pencil Π (on M) is the number

$$\text{rank } \Pi = \max_x \text{rank } \Pi(x) = \max_{\lambda, x} \text{rank } P_\lambda(x).$$

Definition 8. A vector field v is called *bihamiltonian* with respect to a pencil Π , if it is hamiltonian with respect to all brackets of the pencil.

Let Π be a Poisson pencil and v be a vector field which is bihamiltonian with respect to Π . We want to construct a complete family of integrals in involution for v . The main idea for this is provided by the following well-known statement (see [4]).

Proposition 1.2. Let $\Pi = \{P_\lambda\}$ be a Poisson pencil. Then

1. If f is a Casimir function of P_λ for some λ , then f is an integral of any vector field bihamiltonian with respect to Π .
2. If f is a Casimir function of P_λ , g is a Casimir function of P_ν , and $\lambda \neq \nu$, then f and g are in involution with respect to all brackets of the pencil.
3. If f and g are Casimir functions of P_λ , and $\text{rank } P_\lambda(x) = \text{rank } \Pi$ for almost all $x \in M$, then f and g are in involution with respect to all brackets of the pencil.

Let $\hat{\mathcal{F}}$ be the system generated by all Casimir functions of all brackets of the pencil (satisfying the condition $\text{rank } P_\lambda(x) = \text{rank } \Pi$ for almost all $x \in M$). Proposition 1.2 implies that $\hat{\mathcal{F}}$ is a family of integrals of v in involution. However, this system is unsuitable for our purposes by the following two reasons:

1. We can't guarantee that the brackets of the pencil have globally defined Casimir functions.
2. Even if globally defined Casimir functions exist, their behavior may be unpredictable in the neighborhood of the points where the rank of the corresponding bracket drops.

By these reasons we replace the global system $\hat{\mathcal{F}}$ by a local system \mathcal{F} constructed as follows:

Let

$$Bad = \{x \in M : \text{rank } \Pi(x) < \text{rank } \Pi\}.$$

be the set of points in which the rank of all brackets of the pencil drops.

Further we will only consider $x \notin Bad$. For such x we can find α such that $\text{rank } P_\alpha(x) = \text{rank } \Pi$. Moreover, we can find $\varepsilon > 0$ and a neighborhood $U(x)$ such that $\text{rank } P_\nu(y) = \text{rank } \Pi$ for $|\nu - \alpha| < \varepsilon, y \in U(x)$, and all local Casimir functions of P_ν where $|\nu - \alpha| < \varepsilon$ are defined in $U(x)$. Consider a family $\mathcal{F} = \mathcal{F}_{\alpha, \varepsilon}$ generated by all these Casimir functions. Proposition 1.2 implies

Proposition 1.3. *\mathcal{F} is a (local) family of integrals in involution for any system bihamiltonian with respect to our pencil.*

Remark 1.5. The choice of α and ε is not important which means that our results remain true for any choice of α, ε . Moreover, under some additional conditions we will get the same family of integrals for all α, ε . What is important in this construction, is the fact that \mathcal{F} is generated by the Casimir functions of brackets which are **regular** at the point x (see Example 2.4).

1.5 Completeness of \mathcal{F}

Definition 9. The *spectrum* of a pencil Π at a point x is the set

$$\Lambda(x) = \{\lambda \in \overline{\mathbb{C}} : \text{rank } P_\lambda(x) < \text{rank } \Pi(x)\}.$$

Let

$$S = \{x : \Lambda(x) \neq \emptyset\}.$$

Definition 10. We will say that Π is *micro-Kronecker* (or simply *Kronecker*), if the set S has measure zero (i.e. if the spectrum is empty almost everywhere).

In other words, a pencil is Kronecker if its Jordan-Kronecker decomposition has only Kronecker blocks (i.e. has no Jordan blocks) almost everywhere. Giving this definition we follow I. Zakharevich [39] and A. Panasyuk [40].

Theorem 3 (A.V. Bolsinov, [5], the criteria of completeness of \mathcal{F} on a regular symplectic leaf). *Assume that $\text{rank } P_\alpha(x) = \text{rank } \Pi$ and let $O(\alpha, x)$ be the symplectic leaf of P_α passing through x . Then $\mathcal{F}|_{O(\alpha, x)}$ is complete at x if and only if $x \notin S$.*

Corollary 1.1. *\mathcal{F} is complete on $O(\alpha, x)$ if and only if the set $S \cap O(\alpha, x)$ has measure zero.*

Corollary 1.2. *If Π is Kronecker, then \mathcal{F} is complete on almost all regular symplectic leaves.*

The theorem also implies that the singular points of $\mathcal{F}|_{O(\alpha, x)}$ are exactly the points where the rank of some bracket P_β drops. A question arises: **How do we check non-degeneracy of these points and determine their type?**

It turns out that the answer can be given in terms of the so-called linearization of the pencil Π , which will be defined later.

2 Definitions and non-degeneracy criteria

2.1 Linear pencils

Definition 11. Let \mathfrak{g} be a Lie algebra and A be a skew-symmetric bilinear form on it. Then A can be considered as a Poisson tensor on the dual space \mathfrak{g}^* . Assume that the corresponding bracket is compatible with the Lie-Poisson bracket. The Poisson pencil $\Pi^{\mathfrak{g},A} = \{P_\lambda^{\mathfrak{g},A}\}$, where

$$P_\lambda^{\mathfrak{g},A}(x)(\xi, \eta) = \langle x, [\xi, \eta] \rangle + \lambda A(\xi, \eta), \text{ for } \xi, \eta \in \mathfrak{g},$$

will be called the *linear pencil* associated with the pair (\mathfrak{g}, A) .

Giving this definition, we are motivated by the fact that linear pencils arise as a linearization of a general Poisson pencil at a singular point (see Section 2.3). Now we shall discuss some properties of linear pencils. The following is well known (see [27, 41]).

Proposition 2.1. *A form A on \mathfrak{g} is compatible with the Lie-Poisson bracket if and only if this form is a 2-cocycle in terms of the Chevalley-Eilenberg complex, i.e.*

$$dA(\xi, \eta, \zeta) = A([\xi, \eta], \zeta) + A([\eta, \zeta], \xi) + A([\zeta, \xi], \eta) = 0$$

for any $\xi, \eta, \zeta \in \mathfrak{g}$.

Corollary 2.1. *If a form A is compatible with the Lie-Poisson bracket on \mathfrak{g}^* , then $\text{Ker } A$ is a subalgebra in \mathfrak{g} .*

Remark 2.1. The definition of the Chevalley-Eilenberg complex can be found in [42].

Example 2.1 (Exact forms or the “argument shift method”). Let \mathfrak{g} be an arbitrary Lie algebra and $A_a(\xi, \eta) = \langle a, [\xi, \eta] \rangle$, where $a \in \mathfrak{g}^*$. It is easy to see that A_a is compatible with the Lie-Poisson bracket.

The condition $A = A_a$ is equivalent to the fact that $A = da$ in the Chevalley-Eilenberg complex, therefore the compatibility of A_a and the Lie-Poisson bracket follows from the equality $d^2 = 0$.

The pencils with $A = A_a$ are the ones which one should consider when constructing an integrable system by the argument shift method (see [28]), therefore we will call them pencils of the argument shift type. It also seems reasonable to call such linear pencils exact.

We want to use linear pencils to construct commuting functions following the general scheme of Section 1.4. In order to be able to do so, we need the following property of *regularity*:

Definition 12. We will say that a cocycle A on \mathfrak{g} is *regular*, if $\text{rank } \Pi^{\mathfrak{g},A} = \text{rank } A$.

It is easy to see that if $A = A_a$, then $\text{corank } P_\lambda^{\mathfrak{g},A} = \text{ind } \mathfrak{g}$ for any $\lambda \neq \infty$, therefore regularity of A means that $\text{corank } A = \text{ind } \mathfrak{g}$, i.e. it is equivalent to regularity of the element a .

To give a criteria for regularity in the general case, we need a (standard) construction of the central extension of a Lie algebra, associated with a 2-cocycle.

Let \mathfrak{g} be a Lie algebra over a field \mathbb{K} . Suppose that A is a 2-cocycle on \mathfrak{g} . Let us consider the space $\mathfrak{g}_A = \mathfrak{g} + \mathbb{K}^1$, where $\mathbb{K}^1 = \langle z \rangle$ is a one-dimensional vector space, and define a commutator $[\cdot, \cdot]_A$ on \mathfrak{g}_A by the following rule:

$$\begin{aligned} [x, y]_A &= [x, y] + A(x, y)z, \text{ for any } x, y \in \mathfrak{g} \subset \mathfrak{g}_A, \\ [z, \mathfrak{g}_A]_A &= 0. \end{aligned}$$

It is easy to see that if A is closed, then the commutator $[\cdot, \cdot]_A$ turns \mathfrak{g}_A into a Lie algebra. Also note that $\mathfrak{g} = \mathfrak{g}_A / \langle z \rangle$, and the lift of A to \mathfrak{g}_A is an exact form. This means that every closed 2-form on a Lie algebra becomes exact after a lift to a certain one-dimensional central extension.

The following is straightforward

Proposition 2.2. *A 2-cocycle A on \mathfrak{g} is regular if and only if its lift \tilde{A} to \mathfrak{g}_A is regular.*

Corollary 2.2. *If A is a regular cocycle on \mathfrak{g} , then $\text{Ker } A$ is abelian.*

Proof. Let \tilde{A} be the lift of A to \mathfrak{g}_A . Since \tilde{A} is exact and regular, $\text{Ker } \tilde{A}$ is abelian. But $\text{Ker } A = \pi(\text{Ker } \tilde{A})$, where π is the natural projection. Therefore, $\text{Ker } A$ is abelian as well. \square

We will see that in good examples $\text{Ker } A$ is not just an abelian subalgebra, but a Cartan subalgebra.

Suppose that A is regular. Then we can apply the construction of Section 1.4 to this pencil in the neighborhood of the origin (because the origin does not belong to Bad due to regularity of A) and obtain the family \mathcal{F} . This family is involutive with respect to all brackets of the pencil. In particular, with respect to the constant bracket A .

Definition 13. A pencil $\Pi^{\mathfrak{g}, A}$ with regular A will be called *integrable* if \mathcal{F} is complete on the symplectic leaf of A passing through the origin.

We see that if a pencil $\Pi^{\mathfrak{g}, A}$ is integrable, then it canonically defines an integrable system on the symplectic leaf of A passing through the origin.

Proposition 2.3. *A pencil $\Pi^{\mathfrak{g}, A}$ with regular A is integrable if and only if the measure of the set $S \cap O$ is zero where*

$$S = \{x : \text{there exists } \lambda \text{ such that } \text{rank } P_\lambda^{\mathfrak{g}, A}(x) < \text{rank } \Pi^{\mathfrak{g}, A}\}$$

and O is the symplectic leaf of A passing through the origin.

The proof follows from Theorem 3.

Note that if $A = da$, then S is the intersection of \mathfrak{g}^* with a “cylinder” over the set of singular elements in $\mathbb{C} \otimes \mathfrak{g}^*$:

$$S = \mathfrak{g}^* \cap \{x + \lambda a \mid x \text{ is singular in } \mathbb{C} \otimes \mathfrak{g}^*, \lambda \in \mathbb{C}\}.$$

2.2 Singularities associated with integrable linear pencils

Suppose that a pencil $\Pi^{\mathfrak{g}, A}$ is integrable. Then the Casimir functions of the regular brackets of the pencil define an integrable system on the symplectic leaf of A passing through the origin. The origin is a zero-rank singular point for this system. This means that every integrable linear pencil canonically defines a zero-rank singularity (i.e. a germ of an integrable system at singular point). Denote the singularity associated with $\Pi^{\mathfrak{g}, A}$ by $Sing(\Pi^{\mathfrak{g}, A})$.

Example 2.2 (Argument shift on semisimple Lie algebras). Let \mathfrak{g} be a semisimple Lie algebra with two or four-dimensional coadjoint orbits and $A = A_a$ be an “argument shift” form, where $a \in \mathfrak{g}^*$ is a regular element. Below is the list of the corresponding singularities:

1. Two-dimensional orbits: one degree of freedom.
 - $\mathfrak{so}(3)$ - elliptic singularity. See Figure 1.

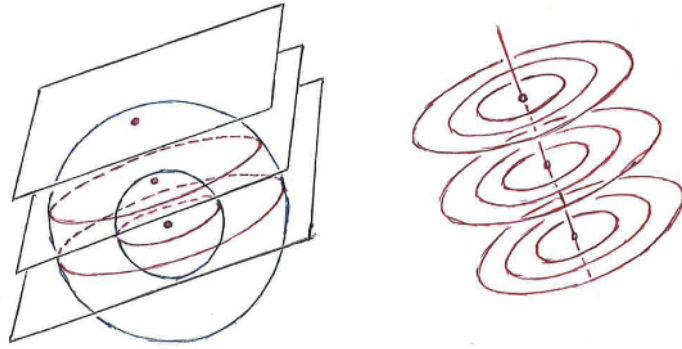


Figure 1: Singularity corresponding to $\mathfrak{so}(3)$

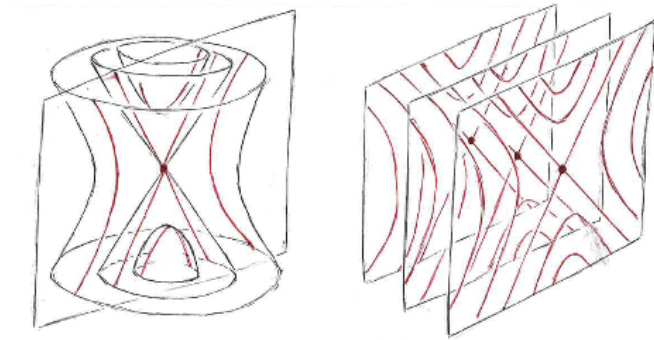


Figure 2: Singularity corresponding to $\mathfrak{sl}(2)$, shift on a hyperbolic element

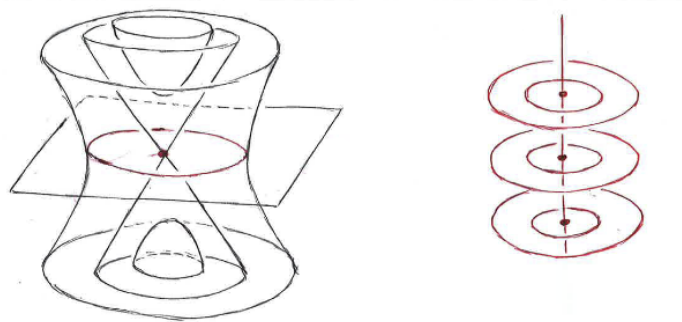


Figure 3: Singularity corresponding to $\mathfrak{sl}(2)$, shift on an elliptic element

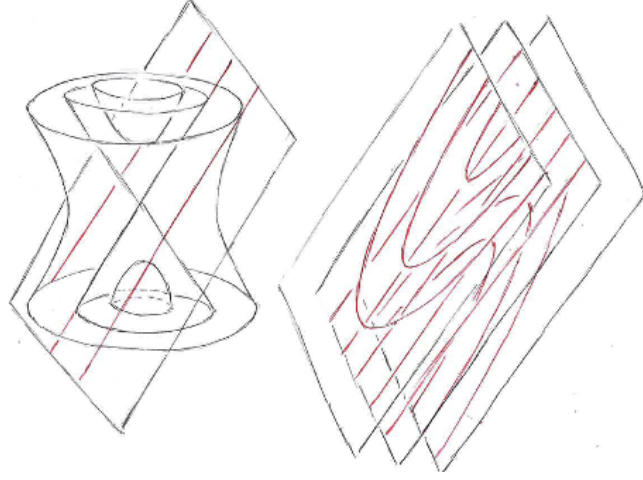


Figure 4: Singularity corresponding to $\mathfrak{sl}(2)$, shift on a nilpotent element

- $\mathfrak{sl}(2)$ - hyperbolic singularity if the Killing form is positive on a (Figure 2), elliptic if it is negative (Figure 3), and degenerate if it is zero (Figure 4).

2. Four-dimensional orbits: two degrees of freedom.

- $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ - center-center singularity (a product of two elliptic singularities).
- $\mathfrak{so}(2, 2) \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ - saddle-saddle (a product of two hyperbolic singularities), saddle-center (a product of an elliptic and a hyperbolic singularity), center-center (a product of two elliptic singularities) or degenerate singularity (depending on a).
- $\mathfrak{so}(3, 1) \simeq \mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})$ - focus-focus singularity if a is semisimple, degenerate otherwise.

We will see further that no semisimple Lie algebras except for the sums of $\mathfrak{so}(3)$, $\mathfrak{sl}(2)$ and $\mathfrak{so}(3, 1)$ give rise to non-degenerate singularities. The corresponding fact in the theory of integrable systems is the Eliasson theorem: all non-degenerate singularities are products of elliptic, hyperbolic and focus-focus singularities (see Theorem 1).

There are also solvable Lie algebras which give rise to non-degenerate singularities.

Example 2.3.

1. Any regular linear pencil on $\mathfrak{e}(2) = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ which is not of argument shift type gives rise to an elliptic singularity. The argument shift pencil gives rise to a degenerate singularity.
2. Any regular linear pencil on $\mathfrak{e}(1, 1) = \mathfrak{so}(1, 1) \ltimes \mathbb{R}^2$ which is not of argument shift type gives rise to a hyperbolic singularity. The argument-shift pencil gives rise to a degenerate singularity.
3. Any regular linear pencil on $\mathfrak{e}(2, \mathbb{C}) \simeq \mathfrak{e}(1, 1, \mathbb{C})$ which is not of argument shift type gives rise to a focus-focus singularity. The argument-shift pencil gives rise to a degenerate singularity.

Definition 14. An integrable linear pencil $\Pi^{\mathfrak{g}, A}$ will be called *non-degenerate*, if the singularity $Sing(\Pi^{\mathfrak{g}, A})$ is non-degenerate.

Suppose that $\Pi^{\mathfrak{g},A}$ is a linear pencil and A is regular. Then $\text{Ker } A \subset \mathfrak{g}$ is an Abelian subalgebra (see Corollary 2.2). Suppose that all elements of $\text{Ker } A$ are ad-semisimple (i.e. $\text{Ker } A$ is a diagonalizable subalgebra). Then \mathfrak{g} admits a “root” decomposition

$$\mathbb{C} \otimes \mathfrak{g} = \mathbb{C} \otimes \text{Ker } A + \sum (V_{\lambda_i} + V_{-\lambda_i}),$$

where $\lambda_i \in (\text{Ker } A)^*$ are “roots” and V_{λ_i} are “root spaces”, which means that for any $\xi \in \mathbb{C} \otimes \text{Ker } A$, $e_{\lambda_i} \in V_{\lambda_i}$ we have $[\xi, e_{\lambda_i}] = \lambda_i(\xi)e_{\lambda_i}$. Roots enter in pairs $\pm\lambda$, because the operators ad_ξ for $\xi \in \text{Ker } A$ belong to $\mathfrak{sp}(\mathfrak{g}/\text{Ker } A, A)$. This follows from Proposition 2.1.

Obviously, the maximal possible number of linear independent λ_i ’s is

$$n = \frac{1}{2}(\dim \mathfrak{g} - \dim \text{Ker } A) = \frac{1}{2}\text{rank } A.$$

Theorem 4. *A linear pencil $\Pi^{\mathfrak{g},A}$ is non-degenerate if and only if $\text{Ker } A$ is a diagonalizable subalgebra and the number of linear independent roots of \mathfrak{g} with respect to $\text{Ker } A$ is exactly $\frac{1}{2}\text{rank } A$. In other words, there exists a decomposition*

$$\mathbb{C} \otimes \mathfrak{g} = \mathbb{C} \otimes \text{Ker } A + \sum (V_{\lambda_i} + V_{-\lambda_i}),$$

where $\lambda_i \in (\text{Ker } A)^*$, all λ_i ’s are linear independent, all $V_{\pm\lambda_i}$ ’s are one-dimensional, and for all $\xi \in \mathbb{C} \otimes \text{Ker } A$, $e_{\lambda_i} \in V_{\lambda_i}$ we have $[\xi, e_{\lambda_i}] = \lambda_i(\xi)e_{\lambda_i}$.

The type of $\text{Sing}(\Pi^{\mathfrak{g},A})$ in this case is (k_e, k_h, k_f) where k_e is the number of pairs of pure imaginary λ_i ’s, k_h is the number of real λ_i ’s, and k_f is the number of pairs of complex conjugate λ_i ’s.

Remark 2.2. We claim that if the conditions of the theorem are satisfied, then A is automatically regular and $\Pi^{\mathfrak{g},A}$ is automatically integrable.

In Section 2.5 we will give an explicit classification of non-degenerate linear pencils. However, it seems that Theorem 4 is more useful for applications, than the classification theorem.

2.3 Linearization of a Poisson pencil

Let P be a Poisson structure on a manifold M , $x \in M$. It is well-known that the linear part of P defines a Lie algebra structure on the kernel of P at the point x . The commutator in this algebra is defined as follows: let $\xi, \eta \in \text{Ker } P(x)$. Choose any functions f, g such that $\text{d}f = \xi, \text{d}g = \eta$, and define

$$[\xi, \eta] = \text{d}\{f, g\}.$$

The following is well-known

Proposition 2.4.

1. The commutator $[\ , \]$ is well-defined and indeed turns $\text{Ker } P(x)$ into a Lie algebra.
2. If $\text{rank } P(x) = \text{rank } P$, then this algebra is Abelian.

Now consider a Poisson pencil $\{P_\lambda = P_0 + \lambda P_\infty\}$ and fix a point $x \notin \text{Bad}$. Denote by $\mathfrak{g}_\lambda(x)$ the Lie algebra on the kernel of P_λ at the point x . For regular λ (i.e. for $\lambda \notin \Lambda(x)$) the algebra \mathfrak{g}_λ is abelian. For singular λ ($\lambda \in \Lambda(x)$) this is not the case in general, therefore $\mathfrak{g}_\lambda(x)$ carries non-trivial information about the behavior of the pencil in the neighborhood of x .

Remark 2.3. For complex values of λ the space $\text{Ker } P_\lambda(x)$ is a subspace of $\mathbb{C} \otimes T_x^*M$, therefore \mathfrak{g}_λ in this case is a complex Lie algebra (see Remark 1.4).

It turns out that apart from the Lie algebra structure \mathfrak{g}_λ carries one more additional structure.

Proposition 2.5. *For any α and β the restrictions of $P_\alpha(x), P_\beta(x)$ on $\mathfrak{g}_\lambda(x)$ coincide up to a multiplicative constant.*

Proof. Since P_λ vanishes on \mathfrak{g}_λ , all other brackets of the pencil are proportional. \square

The restriction $P_\alpha|_{\mathfrak{g}_\lambda}$ is a 2-form on \mathfrak{g}_λ . Therefore, it can be interpreted as a constant Poisson bracket on \mathfrak{g}_λ^* .

Proposition 2.6. *The bracket $P_\alpha|_{\mathfrak{g}_\lambda}$ is compatible with the Lie-Poisson bracket on \mathfrak{g}_λ^* (i.e. $P_\alpha|_{\mathfrak{g}_\lambda}$ is a 2-cocycle on \mathfrak{g}_λ).*

Proof. Since P_α and P_λ are compatible, we have

$$\begin{aligned} & \{\{f, g\}_\alpha, h\}_\lambda + \{\{g, h\}_\alpha, f\}_\lambda + \{\{h, f\}_\alpha, g\}_\lambda + \\ & + \{\{f, g\}_\lambda, h\}_\alpha + \{\{g, h\}_\lambda, f\}_\alpha + \{\{h, f\}_\lambda, g\}_\alpha = 0. \end{aligned}$$

But if $df, dg, dh \in \text{Ker } P_\lambda$ then the first three terms vanish and we can write down

$$\{\{f, g\}_\lambda, h\}_\alpha + \{\{g, h\}_\lambda, f\}_\alpha + \{\{h, f\}_\lambda, g\}_\alpha = 0,$$

or

$$P_\alpha([df, dg], dh) + P_\alpha([dg, dh], df) + P_\alpha([dh, df], dg) = 0,$$

i.e. P_α is a 2-cocycle on \mathfrak{g}_λ , q.e.d. \square

Consequently, $P_\alpha|_{\mathfrak{g}_\lambda}$ defines a linear pencil on \mathfrak{g}_λ^* . Since $P_\alpha|_{\mathfrak{g}_\lambda}$ is defined up to a multiplicative constant, the pencil is well-defined. Denote this pencil by $d_\lambda \Pi(x)$.

Definition 15. The pencil $d_\lambda \Pi(x)$ will be called the λ -linearization of the pencil Π at the point x .

2.4 Non-degeneracy criteria

Definition 16. A pencil Π will be called diagonalizable at a point x , if for each $\lambda \in \Lambda(x)$ and any $\alpha \neq \lambda$ the following is true

$$\dim \text{Ker } (P_\alpha(x)|_{\text{Ker } P_\lambda(x)}) = \text{corank } \Pi(x).$$

Remark 2.4. In terms of the Jordan-Kronecker decomposition for the pencil Π at x this means that all Jordan blocks J_{k_i, λ_i} have size 1×1 , i.e. all k_i are equal to 1.

Theorem 5 (Non-degeneracy criteria). *Let $O(\alpha, x)$ be a symplectic leaf of a bracket P_α passing through x . Assume that P_α is regular at x (or, which is the same, $O(\alpha, x)$ is a symplectic leaf of maximal dimension). Then the singular point x of the integrable system $\mathcal{F}|_{O(\alpha, x)}$ is non-degenerate if and only if the following two conditions hold:*

1. Π is diagonalizable at x .
2. For each $\lambda \in \Lambda(x)$ the linear pencil $d_\lambda \Pi(x)$ is non-degenerate.

The proof of this theorem is given in Section 5.2.

The following example shows that the statement of Theorem 5 is wrong if we add to the system \mathcal{F} a Casimir function of a bracket, singular at x (recall that \mathcal{F} , by definition, is generated by Casimir functions of regular brackets).

Example 2.4. Consider $\mathfrak{so}(3)^*$ with the following bracket

$$P_0 = (x^2 + y^2 + z^2)P_{\mathfrak{so}(3)},$$

where $P_{\mathfrak{so}(3)}$ is the standard Lie-Poisson bracket on $\mathfrak{so}(3)^*$.

Consider also any constant bracket of rank 2 and denote it by P_∞ . It is easy to check that P_0 and P_∞ are compatible.

The Casimir function of P_0 is $x^2 + y^2 + z^2$. The restriction of this function on the symplectic leaf of P_∞ , passing through the origin, defines an integrable system. The origin is a non-degenerate elliptic singular point of this system. However, the linearization of our pencil at the origin is zero, therefore the conditions of Theorem 5 do not hold.

But if we take a Casimir function of a regular bracket, it will look like

$$(x^2 + y^2 + z^2)^2 + \text{linear terms},$$

and its restriction on the symplectic leaf of P_∞ is degenerate, as it is predicted by Theorem 5.

The problem is that if we consider the set of the Casimir functions of all brackets of the pencil, the function $x^2 + y^2 + z^2$ will be an “isolated point” in this set. But if the set of all Casimir functions formed a smooth family, then Theorem 5 could be applied even if \mathcal{F} contained Casimir functions of a bracket singular at a given point (for example, we could take as \mathcal{F} the set of Casimir functions of all brackets). This can be proved by continuity arguments.

Theorem 6 (Type theorem). *Assume that the conditions of Theorem 5 hold. Then the type of the singular point x is (k_e, k_h, k_f) , where*

$$\begin{aligned} k_e &= \sum_{\lambda \in \Lambda(x) \cap \overline{\mathbb{R}}} k_e(\lambda), \\ k_h &= \sum_{\lambda \in \Lambda(x) \cap \overline{\mathbb{R}}} k_h(\lambda), \\ k_f &= \sum_{\lambda \in \Lambda(x) \cap \overline{\mathbb{R}}} k_f(\lambda) + \frac{1}{2} \sum_{\substack{\lambda \in \Lambda(x), \\ \text{Im } \lambda > 0}} (\dim_{\mathbb{C}} \text{Ker } P_\lambda - \text{corank } \Pi), \end{aligned}$$

and $(k_e(\lambda), k_h(\lambda), k_f(\lambda))$ is the type of $\text{Sing}(d_\lambda \Pi(x))$.

In other words, the type of a non-degenerate singular point x is the “sum” of types of $\text{Sing}(d_\lambda \Pi(x))$ for all $\lambda \in \Lambda(x)$. The second summand in the formula for k_f appears, because $\text{Sing}(d_\lambda \Pi(x))$ is always a focus-focus singularity if λ is not real (provided this singularity is non-degenerate).

The proof of Theorem 6 is given in Section 5.2.

Taking into account the Eliasson theorem (Theorem 1), Theorem 6 can be reformulated as follows:

Theorem 7 (Bihamiltonian linearization theorem). *Assume that conditions of Theorem 5 hold. Then the Liouville foliation of the system $\mathcal{F}|_{O(\alpha, x)}$ is locally symplectomorphic to*

$$\left(\prod_{\substack{\lambda \in \Lambda(x), \\ \text{Im } \lambda \geq 0}} \text{Sing}(d_\lambda \Pi(x)) \right) \times (\mathbb{R}^k \times \mathbb{R}^k),$$

where $\mathbb{R}^k \times \mathbb{R}^k$ is a trivial Lagrangian foliation, and k is the rank of x .

In other words, the Liouville foliation of a bihamiltonian system in the neighborhood of a non-degenerate singular point is locally symplectomorphic to the direct product of the Liouville foliations of the λ -linearizations of the system and a trivial foliation.

2.5 Classification of non-degenerate linear pencils

Before giving a classification theorem for non-degenerate linear pencils, we need to define three special algebras.

Definition 17.

1. Denote by \mathfrak{g}_\diamond the Lie algebra generated by e, f, h, t with the following relations:

$$[e, f] = h, [h, \mathfrak{g}_\diamond] = 0, [t, e] = f, [t, f] = -e.$$

This algebra is known as “diamond Lie algebra” (see [43]).

2. Denote by \mathfrak{g}_\diamond^h the Lie algebra generated by e, f, h, t with the following relations:

$$[e, f] = h, [h, \mathfrak{g}_\diamond^h] = 0, [t, e] = e, [t, f] = -f.$$

3. $\mathfrak{g}_\diamond^\mathbb{C} = \mathbb{C} \otimes \mathfrak{g}_\diamond \simeq \mathbb{C} \otimes \mathfrak{g}_\diamond^h$ - the common complexification of these two algebras.

Remark 2.5. The algebras \mathfrak{g}_\diamond and \mathfrak{g}_\diamond^h are (the only non-trivial) one-dimensional central extensions of $\mathfrak{e}(2)$ and $\mathfrak{e}(1, 1)$.

Definition 18. A complex Lie algebra \mathfrak{g} will be called non-degenerate if it can be represented as

$$\mathfrak{g} \simeq \bigoplus \mathfrak{so}(3, \mathbb{C}) \oplus \left(\bigoplus \mathfrak{g}_\diamond^\mathbb{C} \right) / \mathfrak{l}_0 \oplus V,$$

where V is abelian, and \mathfrak{l}_0 is an ideal belonging to the center.

A real Lie algebra \mathfrak{g} will be called non-degenerate if it can be represented as

$$\mathfrak{g} \simeq \bigoplus \mathfrak{so}(3) \oplus \bigoplus \mathfrak{sl}(2) \oplus \bigoplus \mathfrak{so}(3, \mathbb{C}) \oplus \left(\bigoplus \mathfrak{g}_\diamond \oplus \bigoplus \mathfrak{g}_\diamond^h \oplus \bigoplus \mathfrak{g}_\diamond^\mathbb{C} \right) / \mathfrak{l}_0 \oplus V,$$

where V is abelian, and \mathfrak{l}_0 is an ideal belonging to the center⁷.

Let \mathfrak{h} be a Cartan⁸ subalgebra of a non-degenerate Lie algebra \mathfrak{g} . The type of the pair $(\mathfrak{g}, \mathfrak{h})$ is the triple (k_e, k_h, k_f) , where

- k_e = the number of $\mathfrak{so}(3)$ terms in the decomposition of \mathfrak{g} + the number of \mathfrak{g}_\diamond terms + the number of $\mathfrak{sl}(2)$ terms such that the Killing form on $\mathfrak{sl}(2) \cap \mathfrak{h}$ is negative.
- k_h = the number of \mathfrak{g}_\diamond^h terms + the number of $\mathfrak{sl}(2)$ terms such that the Killing form on $\mathfrak{sl}(2) \cap \mathfrak{h}$ is positive.
- k_f = the number of $\mathfrak{so}(3, \mathbb{C})$ terms + the number of $\mathfrak{g}_\diamond^\mathbb{C}$ terms.

Theorem 8 (Classification of non-degenerate linear pencils).

⁷It is possible to show that a real Lie algebra is non-degenerate if its complexification is non-degenerate.

⁸By definition, a Cartan subalgebra of an arbitrary Lie algebra is a nilpotent subalgebra which coincides with its normalizer. In the case of non-degenerate algebras, Cartan subalgebras are simply maximal ad-diagonalizable Abelian subalgebras.

1. A linear pencil $\Pi^{\mathfrak{g},A}$ is non-degenerate if and only if \mathfrak{g} is non-degenerate and $\text{Ker } A$ is a Cartan subalgebra.
2. Assume that $\Pi^{\mathfrak{g},A}$ is non-degenerate. Then the type of $\text{Sing}(\Pi^{\mathfrak{g},A})$ coincides with the type of the pair $(\mathfrak{g}, \text{Ker } A)$.

The proof of the first statement in the complex case can be found in Section 5.3, and in the real case - in Section 5.4. The proof of the second statement is given in Section 5.5.

Remark 2.6. It follows from the theorem that if \mathfrak{g} is non-degenerate and $\text{Ker } A$ is a Cartan subalgebra, then A is automatically regular and the pencil $\Pi^{\mathfrak{g},A}$ is automatically integrable.

Example 2.5. Let $\mathfrak{g} = \mathfrak{g}_{\diamond}$ be the diamond Lie algebra and let $a \in \mathfrak{g}^*$ be such that

$$a(h) = 1, a(e) = a(f) = a(t) = 0.$$

Then Theorem 8 claims that the singularity corresponding to the pencil $\Pi^{\mathfrak{g},da}$ is non-degenerate elliptic. Let us show this explicitly.

The Casimir functions of the Lie-Poisson bracket $\langle x, [\xi, \eta] \rangle$ are given by

$$f_1 = h, f_2 = e^2 + f^2 + 2th.$$

Consequently, the Casimir functions of the “shifted” bracket $\langle x + \lambda a, [\xi, \eta] \rangle$ are given by

$$f_1^\lambda = h + \lambda, f_2^\lambda = e^2 + f^2 + 2t(h + \lambda).$$

The family \mathcal{F} is generated by these functions. The symplectic leaf of A passing through the origin is given by $t = 0, h = 0$. The restriction of the family \mathcal{F} to this leaf is generated by the single function $e^2 + f^2$. Consequently, the corresponding singularity is indeed non-degenerate elliptic.

3 Linear theory

In this section we study properties of two compatible Poisson brackets at a point, i.e. properties of two skew-symmetric bilinear forms on a vector space. Note that all of these properties can be deduced from the Jordan-Kronecker theorem.

Consider a pencil Π and a point x . Define

$$\begin{aligned} \Lambda &= \{\lambda \in \overline{\mathbb{C}} : \text{rank } P_\lambda(x) < \text{rank } \Pi(x)\}. \\ L &= \sum_{\lambda \in \overline{\mathbb{R}} \setminus \Lambda} \text{Ker } P_\lambda(x) \subset T_x^*M. \end{aligned}$$

It is easy to prove the following

Proposition 3.1 (Properties of the space L).

1. The space L is isotropic with respect to any bracket of the pencil.
2. The orthogonal complement to L given by $L^\perp = \{\xi \in T_x^*M \mid P_\alpha(\xi, L) = 0\}$ does not depend on the choice of α .
3. Any regular bracket of the pencil is non-degenerate on L^\perp/L .

4. Let $k = \dim L$. Then for any different $\alpha_1, \dots, \alpha_k \in \overline{\mathbb{R}} \setminus \Lambda$

$$\sum_{i=1}^k \text{Ker } P_{\alpha_i} = L.$$

5. $\dim(\text{Ker } P_\lambda \cap L) = \text{corank } \Pi(x)$ for all λ .

Corollary 3.1. *Let \mathcal{F} be the system of functions defined in Section 1.4. Then $d\mathcal{F} = L$.*

Proof. \mathcal{F} is generated by local Casimir functions of infinite number of regular brackets of the pencil. The differentials of local Casimir functions of a regular bracket generate the kernel of this bracket. Therefore $d\mathcal{F}$ is the sum of kernels of infinite number of regular brackets of the pencil. But it is enough to take $k = \dim L$ of them to generate L (see Proposition 3.1). \square

Since P_β is non-degenerate on L^\perp/L for any regular β , the recursion operator

$$R_\alpha^\beta = P_\beta^{-1} P_\alpha: L^\perp/L \rightarrow L^\perp/L$$

for such β is well-defined.

Proposition 3.2 (Properties of recursion operators).

1. For any α, γ and regular β, δ we can find constants a, b, c, d such that

$$R_\alpha^\beta = (aR_\gamma^\delta + bE)^{-1}(cR_\gamma^\delta + dE),$$

Consequently, the operators R_α^β and R_γ^δ commute and have common eigenspaces. If one of the recursion operators is diagonalizable, then all of them are diagonalizable.

2. Let P_∞ be regular at x . Then the spectrum of the recursion operator R_0^∞ is

$$\sigma(R_0^\infty) = \{-\lambda, \lambda \in \Lambda(x)\}$$

and the λ -eigenspace is

$$\text{Ker}(R_0^\infty - \lambda E) = \text{Ker}(P_{-\lambda} |_{L^\perp/L}).$$

3. The eigenspaces of the recursion operators are pairwise orthogonal with respect to all brackets of the pencil.

4. A pencil is diagonalizable at point x if and only if the corresponding recursion operators are diagonalizable (over \mathbb{C}).

The proof is straightforward.

4 Operator $D_f P$

4.1 Definition of the operator $D_f P$

Non-degeneracy and type are defined in terms of linearizations of hamiltonian vector fields. However, it is more convenient to work with dual operators. In this section we define such dual operator $D_f P$. Let P be a Poisson bracket on M , $x \in M$, $df(x) \in \text{Ker } P(x)$. Define $D_f P(x): T_x^* M \rightarrow T_x^* M$ by the following formula

$$D_f P(x)(\xi) = d\{f, g\}(x),$$

where g is an arbitrary function such that $dg(x) = \xi$.

It is easy to see the following

Proposition 4.1 (Properties of $D_f P$).

1. In local coordinates we have

$$(D_f P(x)(\xi))_k = \frac{\partial P^{ij}}{\partial x^k} \frac{\partial f}{\partial x^i} \xi_j + P^{ij} \frac{\partial^2 f}{\partial x^i \partial x^k} \xi_j$$

and therefore $D_f P(x)(\xi)$ does not depend on the choice of g .

2. $D_f P(x)$ is dual to the linearization of the vector field $\text{sgrad } f = P \text{d}f$ at x .

3. $D_f P(x)$ is skew-symmetric with respect to $P(x)$:

$$P(D_f P(x)(\xi), \eta) + P(\xi, D_f P(x)(\eta)) = 0.$$

4. $\text{Ker } P(x)$ is invariant with respect to $D_f P(x)$.

5. Let $\xi \in \text{Ker } P(x)$. Then

$$D_f P(x)(\xi) = [\text{d}f(x), \xi],$$

where $[,]$ is the commutator in the linearization of P (see Section 2.3).

6. If $\text{rank } P(x) = \text{rank } P$, then $D_f P(x)$ vanishes on $\text{Ker } P(x)$.

4.2 Operators $D_f P_\alpha$ for $f \in \mathcal{F}$

The following lemma will allow us to rewrite the operator $D_f P_\alpha, f \in \mathcal{F}$ as $D_g P_\lambda$ for an appropriate function g .

Lemma 4.1. Let

$$f = \sum_{i=1}^k f_{\alpha_i},$$

where f_{α_i} is a Casimir function of P_{α_i} and $\text{d}f(x) \in \text{Ker } P_\alpha(x)$.

Let $\lambda \in \mathbb{C}$ and $\lambda \neq \alpha_i$ for any i . Consider a function

$$g = \sum_{i=1}^k \frac{\alpha - \alpha_i}{\lambda - \alpha_i} f_{\alpha_i}.$$

Then

1. $\text{d}g(x) \in \text{Ker } P_\lambda$,

2. $D_f P_\alpha(x) = D_g P_\lambda(x)$.

The proof is a straightforward computation.

Corollary 4.1. Let $f \in \mathcal{F}, \text{d}f(x) \in \text{Ker } P_\alpha$. Then $D_f P_\alpha$ is skew-symmetric with respect to **all** brackets of the pencil.

Proof. It is enough to show that $D_f P_\alpha$ is skew-symmetric with respect to two brackets of the pencil. By Proposition 4.1 it is skew-symmetric with respect to P_α . By Lemma 4.1 we can find $\beta \neq \alpha$ and a function g such that $D_f P_\alpha = D_g P_\beta$. Therefore $D_f P_\alpha$ is skew-symmetric with respect to P_β as well, q.e.d. \square

Proposition 4.2. *Let $f \in \mathcal{F}$, $df(x) \in \text{Ker } P_\alpha$. Then $D_f P_\alpha$ vanishes on L .*

Proof. By Lemma 4.1 for almost all β we can find a function g such that $D_f P_\alpha = D_g P_\beta$. If β is regular, the operator $D_g P_\beta$ vanishes on $\text{Ker } P_\beta$ (see Proposition 4.1), therefore $D_f P_\alpha$ vanishes on $\text{Ker } P_\beta$ for almost all regular β . But it is enough to take finite number of values of β to generate L (see Proposition 3.1), therefore $D_f P_\alpha$ vanishes on L . \square

Consequently, the operators $D_f P_\alpha$ are well-defined on the space L^\perp/L .

Proposition 4.3. *Let $f \in \mathcal{F}$, $df(x) \in \text{Ker } P_\alpha$. Then the operator $D_f P_\alpha|_{L^\perp/L}$ has the following properties:*

1. *Belongs to $\mathfrak{sp}(L^\perp/L, P_\beta)$ for any regular β , i.e. is bi-symplectic.*
2. *Commutates with the recursion operators.*
3. *Preserves common eigenspaces of the recursion operators.*
4. *If α is regular, then there exists $\tilde{f} \in \mathcal{F}$ such that $D_f P_\alpha|_{L^\perp/L} R_\beta^\alpha = D_{\tilde{f}} P_\alpha|_{L^\perp/L}$.*

Proof. It is enough to prove statement (4). Let f_α be a Casimir function of P_α such that $df_\alpha = df$. Then $d(f - f_\alpha) = 0$ and

$$D_{f-f_\alpha} P_\alpha|_{L^\perp/L} = D_f P_\alpha|_{L^\perp/L}.$$

On the other hand, by Proposition 4.1,

$$D_{f-f_\alpha} P_\alpha|_{L^\perp/L} = d^2(f - f_\alpha) P_\alpha.$$

Therefore,

$$D_f P_\alpha|_{L^\perp/L} R_\beta^\alpha = d^2(f - f_\alpha) P_\alpha R_\beta^\alpha = d^2(f - f_\alpha) P_\beta = D_{f-f_\alpha} P_\beta|_{L^\perp/L}.$$

Now let $\tilde{\alpha} \rightarrow \alpha$ and let $f_{\tilde{\alpha}}$ be a Casimir function of $P_{\tilde{\alpha}}$ depending smoothly on $\tilde{\alpha}$. By lemma 4.1 we can write

$$D_{f-f_{\tilde{\alpha}}} P_\beta|_{L^\perp/L} = D_{g(\tilde{\alpha})} P_\alpha|_{L^\perp/L}$$

for some function $g(\tilde{\alpha}) \in \mathcal{F}$. Obviously,

$$\lim_{\tilde{\alpha} \rightarrow \alpha} D_{f-f_{\tilde{\alpha}}} P_\beta|_{L^\perp/L} = D_{f-f_\alpha} P_\beta|_{L^\perp/L},$$

therefore $D_{f-f_\alpha} P_\beta|_{L^\perp/L}$ belongs to the closure of $\{D_h P_\alpha|_{L^\perp/L}\}_{h \in \mathcal{F}}$. But this latter space is finite-dimensional and, therefore, closed, which proves our proposition. \square

The following Proposition allows us to calculate $D_f P_\alpha$ on an eigenspace of a recursion operator.

Proposition 4.4.

1. *Let $f = \sum_{i=1}^k f_{\alpha_i}$, where f_{α_i} is a Casimir function of a regular bracket P_{α_i} . Let also $df(x) \in \text{Ker } P_\alpha$, $\lambda \in \Lambda(x)$. Then $D_f P_\alpha|_{\text{Ker } P_\lambda} = \text{ad}_\xi$, where*

$$\xi = \sum_{i=1}^k \frac{\alpha - \alpha_i}{\lambda - \alpha_i} df_{\alpha_i}$$

and ad_ξ is the adjoint operator in \mathfrak{g}_λ .

2. The following sets of operators are equal

$$\{D_f P_\alpha|_{\text{Ker } P_\alpha}\}_{f \in \mathcal{F}, df \in \text{Ker } P_\alpha} = \{\text{ad}_\xi\}_{\xi \in \mathfrak{g}_\lambda \cap L},$$

where ad_ξ is the adjoint operator in \mathfrak{g}_λ .

Proof. This directly follows from Proposition 4.1 and Lemma 4.1. \square

Now note that if the recursion operators are diagonalizable, we are able to express $D_f P_\alpha$ on the whole L^\perp/L via adjoint operators. Indeed, L^\perp/L in this case is going to be the direct sum of $\text{Ker } P_\lambda|_{L^\perp/L}$, $\lambda \in \Lambda(x)$.

4.3 Operator $D_f P$ and linearizations of hamiltonian vector fields

Let us consider the integrable system $\mathcal{F}|_{O(\alpha, x)}$ defined in Section 1.4. We assume that the symplectic leaf $O(\alpha, x)$ is regular.

The tangent space $T_x O$ is equipped with a natural symplectic form ω_α given by the formula

$$\omega_\alpha(P_\alpha df, P_\alpha dg) = \{f, g\}_\alpha(x).$$

Let

$$W = \{\text{sgrad}_\alpha f(x) = P_\alpha df(x)\}_{f \in \mathcal{F}}.$$

Let W^\perp be the orthogonal complement to it (with respect to ω_α). Then the space W^\perp/W is symplectic with respect to ω_α .

Proposition 4.5. *Consider the map $P_\alpha : T_x^* M \rightarrow T_x M$. The following is true:*

1. $\omega_\alpha(P_\alpha(\xi), P_\alpha(\eta)) = P_\alpha(\xi, \eta)$.
2. $P_\alpha(L) = W$.
3. $P_\alpha(L^\perp) = W^\perp$.
4. Let A_f be the linearization of $\text{sgrad } f$ on W^\perp/W , where $df \in \text{Ker } P_\alpha \cap L$. Then the following diagram is commutative:

$$\begin{array}{ccc} L^\perp/L & \xrightarrow{D_f P_\alpha} & L^\perp/L \\ \downarrow P_\alpha & & \downarrow P_\alpha \\ W^\perp/W & \xrightarrow{A_f} & W^\perp/W \end{array}$$

The proof is straightforward.

Corollary 4.2. P_α defines a symplectomorphism between L^\perp/L and W^\perp/W . This symplectomorphism sends $D_f P_\alpha$ to A_f , which is the linearization of $\text{sgrad } f$.

Corollary 4.3. Singular point x is non-degenerate on the regular symplectic leaf of P_α passing through x if and only if the set of operators

$$\{D_f P_\alpha|_{L^\perp/L}\}_{f \in \mathcal{F}, df \in \text{Ker } P_\alpha}$$

generate a Cartan subalgebra in $\mathfrak{sp}(L^\perp/L, P_\alpha)$. Type of the point x coincides with the type of this Cartan subalgebra.

Proof. By definition x is non-degenerate if and only if the linearizations of the hamiltonian vector fields $\text{sgrad } f$, $f \in \mathcal{F}$ generate a Cartan subalgebra in $\mathfrak{sp}(W^\perp/W, \omega_\alpha)$. Now we need to apply the isomorphism constructed above. \square

5 Proof of the main theorems

5.1 Proof of Theorem 4

Let $\Pi^{\mathfrak{g},A}$ be an integrable linear pencil. Construct the system \mathcal{F} (see Section 1.4) for this pencil and consider the singular point 0 on the orbit of the regular bracket A . By Corollary 4.3, to check non-degeneracy and find the type of this point, we need to calculate operators $D_f A$ on L^\perp/L , for $f \in F, df \in \text{Ker } A$.

By Proposition 4.4 we have

$$\{D_f A|_{\text{Ker } P_0}\}_{f \in \mathcal{F}, df \in \text{Ker } A} = \{\text{ad}_\xi\}_{\xi \in \text{Ker } P_0 \cap L},$$

where P_0 is the Lie-Poisson bracket. But $\text{Ker } P_0 = \mathfrak{g}^*$, while $\text{Ker } A = L$, therefore

$$\{D_f A\}_{f \in \mathcal{F}, df \in \text{Ker } A} = \{\text{ad}_\xi\}_{\xi \in \text{Ker } A}.$$

Since $L^\perp = \mathfrak{g}^*$, we have

$$\{D_f A|_{L^\perp/L}\}_{f \in \mathcal{F}, df \in \text{Ker } A} = \{\text{ad}_\xi|_{\mathfrak{g}/\text{Ker } A}\}_{\xi \in \text{Ker } A}.$$

Taking into account Corollary 4.3, this proves the following

Lemma 5.1. *An integrable linear pencil $\Pi^{\mathfrak{g},A}$ is non-degenerate if and only if the set of operators*

$$\{\text{ad}_\xi|_{\mathfrak{g}/\text{Ker } A}\}_{\xi \in \text{Ker } A}$$

is a Cartan subalgebra in $\mathfrak{sp}(\mathfrak{g}/\text{Ker } A, A)$. The type of $\text{Sing}(\Pi^{\mathfrak{g},A})$ coincides with the type of this subalgebra.

Remark 5.1. Since A is a skew-symmetric 2-form, the space $\mathfrak{g}/\text{Ker } A$ is symplectic. The condition of compatibility of A with the Lie-Poisson bracket implies that all operators ad_ξ , where $\xi \in \text{Ker } A$, are skew-symmetric with respect to A . Therefore, they generate an Abelian (see Corollary 2.2) subalgebra in $\mathfrak{sp}(\mathfrak{g}/\text{Ker } A, A)$. Now we see that non-degeneracy is equivalent to the fact that this subalgebra is a Cartan subalgebra.

Corollary 5.1. *If $\Pi^{\mathfrak{g},A}$ is non-degenerate, then $\text{Ker } A$ consists of ad-semisimple elements.*

Since $\text{Ker } A$ is a commutative subalgebra (see Corollary 2.2) which consists of semisimple elements, all operators $\text{ad}_\xi, \xi \in \text{Ker } A$ may be simultaneously diagonalized (over \mathbb{C}). Now we can consider the “root” decomposition of \mathfrak{g} :

$$\mathbb{C} \otimes \mathfrak{g} = \text{Ker } A + \sum_{i=1}^n (V_{\lambda_i} + V_{-\lambda_i}),$$

where each $V_{\pm\lambda_i}$ is spanned by one common eigenvector corresponding to the eigenvalue $\pm\lambda(\xi)$. Eigenvalues enter in pairs because the operators ad_ξ are symplectic.

Remark 5.2. Note that all $V_{\pm\lambda_i}$ are one-dimensional by definition.

Proposition 5.1. *If $\text{Ker } A$ is diagonalizable, then the pencil is non-degenerate if and only if $\lambda_1, \dots, \lambda_n$ are linearly independent as linear functions on $\text{Ker } A$. Type of $\text{Sing}(\Pi^{\mathfrak{g},A})$ is (k_e, k_h, k_f) where k_e is the number of pure imaginary λ_i , k_h is the number of real λ_i , and k_f is the number of pairs of complex conjugate λ_i .*

Proof. Consider the map

$$\mathrm{ad}: \mathrm{Ker} A \rightarrow \mathfrak{sp}(\mathfrak{g}/\mathrm{Ker} A, A),$$

which sends ξ to the operator ad_ξ . We want the image of this map to be a Cartan subalgebra (see Lemma 5.1). Since it is abelian and diagonalizable, it is Cartan if and only if the dimension of it equals

$$n = \frac{1}{2} \dim \mathfrak{g}/\mathrm{Ker} A.$$

Obviously,

$$\dim \mathrm{ad}(\mathrm{Ker} A) = \dim_{\mathbb{C}} \langle \lambda_i \rangle,$$

where $\langle \lambda_i \rangle$ is the subspace in $(\mathrm{Ker} A)_{\mathbb{C}}^*$ spanned by $\lambda_1, \dots, \lambda_n$. Therefore, $\mathrm{ad}(\mathrm{Ker} A)$ is a Cartan subalgebra if and only if the roots are linearly independent.

The second statement directly follows from the definition of type. \square

Corollary 5.2. *If $\Pi^{\mathfrak{g}, A}$ is non-degenerate, then $\mathrm{Ker} A$ is a Cartan subalgebra.*

Proof of Theorem 4. Taking into account Corollary 5.1 and Proposition 5.1, it suffices to show that A is regular and $\Pi^{\mathfrak{g}, A}$ is integrable. In Sections 5.3, 5.4 we will see that the conditions of the theorem imply that \mathfrak{g} is a non-degenerate Lie algebra. Therefore, the central extension $\tilde{\mathfrak{g}}_A$ is non-degenerate as well. Since $\mathrm{Ker} A$ is a Cartan subalgebra in \mathfrak{g} , the kernel of the lift \tilde{A} of A to $\tilde{\mathfrak{g}}_A$ is also a Cartan subalgebra. It is easy to check that for a non-degenerate Lie algebra the dimension of a Cartan subalgebra equals the index. Therefore, \tilde{A} is regular on $\tilde{\mathfrak{g}}_A$. Now Proposition 2.2 implies regularity of A .

Now note that we do not need to prove integrability, because it automatically follows from non-degeneracy at the origin. Indeed, we can always find a regular point in the neighborhood of a non-degenerate point. Due to analyticity regular points are everywhere dense, q.e.d. \square

5.2 Proof of Theorems 5, 6

By Corollary 4.3, a singular point x is non-degenerate on a regular symplectic leaf of a bracket P_α if and only if the operators $D_f P_\alpha$, where $f \in \mathcal{F}$, $df \in \mathrm{Ker} P_\alpha$, generate a Cartan subalgebra in $\mathfrak{sp}(L^\perp/L, P_\alpha)$. The type of the singular point coincides with the type of this subalgebra.

Proposition 5.2. *Suppose that a point x is non-degenerate. Then the pencil Π is diagonalizable at the point x .*

Proof. Indeed, since $D_f P_\alpha$ generate a Cartan subalgebra in $\mathfrak{sp}(L^\perp/L, P_\alpha)$, they should be diagonalizable. Moreover, we can find a linear combination of these operators, which have distinct eigenvalues. The recursion operator must commute with this linear combination (by Proposition 4.3). Therefore, the recursion operator must be diagonalizable. Now it suffices to apply Proposition 3.2. \square

In the diagonalizable situation the space L^\perp/L is decomposed, together with the form P_α , into the direct sum of the eigenspaces of the recursion operators (Proposition 3.2). These eigenspaces are invariant with respect to the operators $D_f P_\alpha$ (Proposition 4.3).

Proposition 5.3. *Denote*

$$\begin{aligned} D_\alpha &= \{D_f P_\alpha \mid L^\perp/L\}_{f \in \mathcal{F}, df \in \mathrm{Ker} P_\alpha} \subset \mathfrak{sp}(L^\perp/L, P_\alpha), \\ D_{\alpha, \lambda} &= \{D_f P_\alpha \mid K_\lambda\}_{f \in \mathcal{F}, df \in \mathrm{Ker} P_\alpha} \subset \mathfrak{sp}(K_\lambda, P_\alpha), \end{aligned}$$

where $K_\lambda = \text{Re Ker } P_\lambda |_{\mathbb{C} \otimes L^\perp / L}$. Then, provided the pencil is diagonalizable,

$$D_\alpha = \bigoplus_{\substack{\lambda \in \Lambda(x), \\ \text{Im } \lambda \geq 0}} D_{\alpha, \lambda}.$$

Proof. It is enough to note that D_α is invariant under the multiplication by a recursion operator (Proposition 4.3). \square

Proposition 5.4. *Let Π be a pencil diagonalizable at x . Then the singular point x is non-degenerate on a regular symplectic leaf of bracket P_α if and only if for each $\lambda \in \Lambda(x)$ the set of operators $D_f P_\alpha$ generate a Cartan subalgebra in*

$$\mathfrak{sp}(\text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}), P_\alpha).$$

The type of x is the sum of types of these Cartan subalgebras.

Remark 5.3. For real λ

$$\text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}) = \text{Ker } (P_\lambda |_{L^\perp / L}).$$

Proof. This follows from Proposition 5.3 and Corollary 4.3. \square

Proposition 5.5. *Let Π be a pencil diagonalizable at x . Let $\mathbb{K} = \mathbb{R}$ if λ is real and \mathbb{C} otherwise. Then the set of operators $D_f P_\alpha$ generate a Cartan subalgebra in $\mathfrak{sp}(\text{Ker } (P_\lambda |_{\mathbb{K} \otimes L^\perp / L}), P_\alpha)$ if and only if the pencil $d_\lambda \Pi(x)$ is non-degenerate. The type of this subalgebra for real λ coincides with the type of $\text{Sing}(d_\lambda \Pi(x))$.*

Proof. By Proposition 4.4

$$\{D_f P_\alpha |_{\text{Ker } P_\lambda}, f \in \mathcal{F}, df \in \text{Ker } P_\alpha\} = \{\text{ad}_\xi, \xi \in \mathfrak{g}_\lambda \cap \mathbb{K} \otimes L\}.$$

In the diagonalizable case we have $\mathfrak{g}_\lambda \cap \mathbb{K} \otimes L = \text{Ker } (P_\alpha |_{\mathfrak{g}_\lambda})$.

But

$$\mathfrak{h} = \{\text{ad}_\xi, \xi \in \text{Ker } (P_\alpha |_{\mathfrak{g}_\lambda})\}$$

is a Cartan subalgebra in $\mathfrak{sp}(\mathfrak{g}_\lambda / \mathfrak{h})$ if and only if the pencil $d_\lambda \Pi(x)$ is non-degenerate (Proposition 5.1). The type of this subalgebra coincides with the type of $\text{Sing}(d_\lambda \Pi(x))$ by the same Proposition 5.1. \square

Proposition 5.6. *For a complex value of λ the set of operators $D_f P_\alpha$ generate a Cartan subalgebra in $\mathfrak{sp}(\text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}), P_\alpha)$ if and only if the pencil $d_\lambda \Pi(x)$ is non-degenerate. The type of this subalgebra is $(0, 0, k_f)$, where k_f equals half of its dimension.*

Proof. Let the pencil $d_\lambda \Pi(x)$ be non-degenerate. Then the pencil $d_{\bar{\lambda}} \Pi(x)$ is also non-degenerate. But this means that the set of operators $D_f P_\alpha$ generate a Cartan subalgebra in

$$\mathfrak{sp}(\text{Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}) \oplus \text{Ker } (P_{\bar{\lambda}} |_{\mathbb{C} \otimes L^\perp / L}), P_\alpha).$$

Now note that

$$(\text{Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}) \oplus \text{Ker } (P_{\bar{\lambda}} |_{\mathbb{C} \otimes L^\perp / L})) \cap (L^\perp / L) = \text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}).$$

Therefore operators $D_f P_\alpha$ generate a Cartan subalgebra in $\mathfrak{sp}(\text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L}), P_\alpha)$ as well (taking into account that these operators are real). The inverse statement is proved analogously.

To prove the statement about the type note that if $D_f P_\alpha$ has pure imaginary or real eigenvalue on $\text{Re Ker } (P_\lambda |_{\mathbb{C} \otimes L^\perp / L})$, then ad_ξ in \mathfrak{g}_λ has the same eigenvalue. But a generic element in Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ doesn't have such eigenvalues. \square

Proof of Theorem 5. We have already shown that it is necessary for a pencil to be diagonalizable. Therefore it suffices to show that for a diagonalizable pencil x is non-degenerate if and only if for each $\lambda \in \Lambda(x)$ the linear pencil $d_\lambda \Pi(x)$ is non-degenerate. But taking into account Propositions 5.5, 5.6, our statement directly follows from Proposition 5.4. \square

Proof of Theorem 6. Taking into account Propositions 5.5, 5.6, our statement directly follows from Proposition 5.4. \square

5.3 Proof of Theorem 8: the complex case

$\Pi^{\mathfrak{g}, A}$ is non-degenerate $\Rightarrow \mathfrak{g}$ is non-degenerate, and $\text{Ker } A$ is a Cartan subalgebra.

Taking into account Theorem 4 and Corollary 5.2, it suffices to show that if a complex Lie algebra \mathfrak{g} admits a root decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{i=1}^n (V_{\lambda_i} + V_{-\lambda_i}), \quad (2)$$

with linearly independent λ_i , then \mathfrak{g} is non-degenerate.

By definition we have

$$\begin{aligned} [\mathfrak{h}, \mathfrak{h}] &= 0, \\ [h, x] &= \lambda(h)x \text{ for } h \in \mathfrak{h}, x \in V_\lambda. \end{aligned}$$

The following is standard

Proposition 5.7. *If $e_\alpha \in V_\alpha, e_\beta \in V_\beta$, then $[e_\alpha, e_\beta] \in V_{\alpha+\beta}$.*

Since the roots are independent, $\alpha + \beta$ is a root if and only if $\beta = -\alpha$. Consequently, we have the following relations

$$\begin{aligned} [V_{\lambda_i}, V_{-\lambda_i}] &\in \mathfrak{h}, \\ [V_{\lambda_i}, V_{\pm\lambda_j}] &= 0 \text{ for } i \neq j. \end{aligned}$$

Let e_i be a basis vector in V_{λ_i} and e_{-i} be a basis vector in $V_{-\lambda_i}$. Denoting $h_i = [e_i, e_{-i}]$, we will have

$$[h_i, e_j] = [[e_i, e_{-i}], e_j] = 0$$

if $i \neq j$ due to the Jacobi identity. Therefore,

$$\lambda_j(h_i) = 0 \text{ for } i \neq j.$$

Now suppose that $\lambda_i(h_i) \neq 0$ for some value of i . Then the triple e_i, e_{-i}, h_i generates a subalgebra isomorphic to $\mathfrak{so}(3, \mathbb{C})$. We claim that it admits a complementary subalgebra in \mathfrak{g} . Let

$$\tilde{\mathfrak{h}} = \{h \in \mathfrak{h} : \lambda_i(h) = 0\}.$$

Denote

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} + \sum_{j \neq i} (V_{\lambda_j} + V_{-\lambda_j}).$$

Proposition 5.8. $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \langle e_i, e_{-i}, h_i \rangle$.

The proof is straightforward.

After separating $\mathfrak{so}(3)$ summands for all i such that $\lambda_i(h_i) \neq 0$, we may assume that $\lambda_i(h_i) = 0$ for all i .

Now we shall separate an abelian summand. Decompose the center of \mathfrak{g} into a direct sum

$$Z(\mathfrak{g}) = (Z(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]) \oplus V.$$

It is obvious that V can be separated from \mathfrak{g} as a direct summand.

After separating the abelian summand we may assume that

$$Z(\mathfrak{g}) \subset [\mathfrak{g}, \mathfrak{g}].$$

This means that the center is generated by $h_i = [e_i, e_{-i}]$.

Now decompose subalgebra \mathfrak{h} as follows

$$\mathfrak{h} = \langle h_1, \dots, h_k \rangle \oplus T.$$

Since $\lambda_i(h_j) = 0$ for all i and j , all λ_i are linearly independent in T^* . Moreover, for each $t \in T$ there is i such that $\lambda_i(t) \neq 0$ (otherwise t belongs to the center, which is not possible, because the center is generated by h_i). Therefore, the set of λ_i is a basis in T^* and we can choose a basis t_1, \dots, t_k in T such that

$$\lambda_i(t_j) = \delta_{ij}.$$

Consequently, \mathfrak{g} is generated by e_i, e_{-i}, h_i, t_i with the following relations

$$\begin{aligned} [e_i, e_{-i}] &= h_i, \\ [e_i, e_j] &= 0 \text{ if } j \neq -i, \\ [t_i, e_i] &= e_i, \\ [t_i, e_{-i}] &= -e_{-i}, \\ [t_i, e_j] &= 0 \text{ if } j \neq \pm i, \\ [h_i, \mathfrak{g}] &= 0, \\ [t_i, t_j] &= 0. \end{aligned}$$

If all h_i were linearly independent, our algebra could be decomposed into the direct sum of $\mathfrak{g}_{\diamond}^{\mathbb{C}}$ -subalgebras. Since this is not necessarily the case, \mathfrak{g} is a quotient of such a direct sum by some central ideal.

Therefore, \mathfrak{g} is indeed non-degenerate.

\mathfrak{g} is non-degenerate and $\text{Ker } A$ is a Cartan subalgebra $\Rightarrow \Pi^{\mathfrak{g}, A}$ is regular, integrable and non-degenerate.

It is enough to apply Theorem 4.

5.4 Proof of Theorem 8: the real case

It is possible to classify real non-degenerate linear pencils by classifying the real forms of complex non-degenerate algebras. However it seems to be better for the logic of the text to do it explicitly.

$\Pi^{\mathfrak{g}, A}$ is non-degenerate $\Rightarrow \mathfrak{g}$ is non-degenerate and $\text{Ker } A$ is a Cartan subalgebra.

By Theorem 4 and Corollary 5.2 the subalgebra $\mathfrak{h} = \text{Ker } A$ is a Cartan subalgebra and the roots are linearly independent. These roots have the form $\pm\lambda_1, \dots, \pm\lambda_k, \pm\nu_1 i, \dots, \pm\nu_l i, \pm\mu_1 \pm \xi_1 i, \dots, \pm\mu_m \pm \xi_m i$. We can write

$$\mathfrak{g} = \mathfrak{h} + \langle e_{\pm 1}, \dots, e_{\pm k}, f_{\pm 1}, \dots, f_{\pm l}, g_{\pm 1}, h_{\pm 1}, \dots, g_{\pm m}, h_{\pm m} \rangle,$$

where

$$\begin{aligned} [x, e_i] &= \lambda_i(x)e_i \text{ for } x \in \mathfrak{h}, \\ [x, f_i] &= \nu_i(x)f_{-i} \text{ for } x \in \mathfrak{h}, \\ [x, g_i] &= \mu_i(x)g_i - \xi_i(x)h_i \text{ for } x \in \mathfrak{h}, \\ [x, h_i] &= \xi_i(x)g_i + \mu_i(x)h_i \text{ for } x \in \mathfrak{h}, \end{aligned}$$

where, by definition,

$$\begin{aligned} \lambda_{-i} &= -\lambda_i, \nu_{-i} = -\nu_i, \\ \mu_{-i} &= -\mu_i, \xi_{-i} = -\xi_i. \end{aligned}$$

It is easy to check that

$$\begin{aligned} [e_i, e_{-i}] &\in \mathfrak{h}, \\ [f_i, f_{-i}] &\in \mathfrak{h}, \\ [g_i, g_{-i}] &= -[h_i, h_{-i}] \in \mathfrak{h}, \\ [g_i, h_{-i}] &= [h_i, g_{-i}] \in \mathfrak{h}, \end{aligned}$$

and all other commutators vanish.

Suppose that $\lambda_i([e_i, e_{-i}]) \neq 0$ for some i . In this case the triple $e_i, e_{-i}, [e_i, e_{-i}]$ generate a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. It can be shown that this subalgebra can be separated as a direct summand, analogous to the complex case.

Similarly, if $\nu_i([f_i, f_{-i}]) \neq 0$, we obtain a summand isomorphic to $\mathfrak{so}(3, \mathbb{R})$ if $\nu_i([f_i, f_{-i}]) > 0$ and $\mathfrak{sl}(2, \mathbb{R})$ if $\nu_i([f_i, f_{-i}]) < 0$.

Further, we have

$$\begin{aligned} \xi_i([g_i, g_{-i}])g_i + \mu_i([g_i, g_{-i}])h_i &= [[g_i, g_{-i}], h_i] = \\ &= -[[g_{-i}, h_i], g_i] = \mu_i([g_{-i}, h_i])g_i - \xi_i([g_{-i}, h_i])h_i, \end{aligned}$$

therefore

$$\begin{aligned} \xi_i([g_i, g_{-i}]) &= \mu_i([g_{-i}, h_i]), \\ \mu_i([g_i, g_{-i}]) &= -\xi_i([g_{-i}, h_i]), \end{aligned}$$

which means that ξ_i and ν_i are either linearly independent on the subspace $\langle [g_i, g_{-i}], [g_{-i}, h_i] \rangle$ or both vanish. In the first case the elements $g_i, g_{-i}, h_i, h_{-i}, [g_i, g_{-i}], [g_{-i}, h_i]$ generate a subalgebra isomorphic to $\mathfrak{so}(3, \mathbb{C})$.

After separating all described summands, we see that

$$\begin{aligned} [e_i, e_{-i}] &\in Z(\mathfrak{g}), \\ [f_i, f_{-i}] &\in Z(\mathfrak{g}), \\ [g_i, g_{-i}] &= -[h_i, h_{-i}] \in Z(\mathfrak{g}), \\ [g_i, h_{-i}] &= [h_i, g_{-i}] \in Z(\mathfrak{g}). \end{aligned}$$

In the way absolutely similar to the complex case it can be shown that \mathfrak{g} is decomposed into a sum of an abelian algebra and a quotient of a sum of several copies of $\mathfrak{g}_{\diamond}^h, \mathfrak{g}_{\diamond}, \mathfrak{g}_{\diamond}^{\mathbb{C}}$ by some central ideal.

\mathfrak{g} is non-degenerate and $\text{Ker } A$ is a Cartan subalgebra $\Rightarrow \Pi^{\mathfrak{g}, A}$ is regular, integrable and non-degenerate.

It is enough to apply Theorem 4.

5.5 Proof of the second part of Theorem 8

In this section we will keep the notations of Section 5.4.

By Proposition 5.1 the type of $Sing(\Pi^{\mathfrak{g},A})$ is (k_e, k_h, k_f) , where k_e is the number of pairs of roots of type $\pm\nu_j i$, k_h is the number of pairs of type $\pm\lambda_j$, k_f is the number of quadruples of type $\pm\mu_j \pm \xi_j i$.

Now note that in the proof of Theorem 8 only complex roots gave rise to summands of type $\mathfrak{so}(3, \mathbb{C}), \mathfrak{g}_{\diamond}^{\mathbb{C}}$. This means that k_f indeed coincides with the number of summands of type $\mathfrak{so}(3, \mathbb{C}), \mathfrak{g}_{\diamond}^{\mathbb{C}}$.

For a pair of pure imaginary roots there are three possibilities:

1. $\nu_j([f_j, f_{-j}]) = 0 \Rightarrow \mathfrak{g}_{\diamond}$.
2. $\nu_j([f_j, f_{-j}]) < 0 \Rightarrow \mathfrak{sl}(2, \mathbb{R})$.
3. $\nu_j([f_j, f_{-j}]) > 0 \Rightarrow \mathfrak{so}(3, \mathbb{R})$.

Let us consider the second case and calculate the Killing form on the element $z = [f_j, f_{-j}]$. We have

$$\begin{aligned} [z, f_j] &= \nu_j(z)f_{-j}, \\ [z, f_{-j}] &= -\nu_j(z)f_j. \end{aligned}$$

The value of the Killing form on the element z is equal to $\text{tr}(\text{ad}z)^2 = -2\nu_i(z)^2 < 0$.

Now let us consider the case of a pair of real roots. There are two possibilities:

1. $\lambda_j([e_j, e_{-j}]) = 0 \Rightarrow \mathfrak{g}_{\diamond}^h$.
2. $\lambda_j([e_j, e_{-j}]) \neq 0 \Rightarrow \mathfrak{sl}(2, \mathbb{R})$.

Consider the second case and calculate the Killing form on $z = [e_j, e_{-j}]$. We have

$$\begin{aligned} [z, e_j] &= \lambda_j(z)e_j, \\ [z, e_{-j}] &= -\lambda_j(z)e_{-j}. \end{aligned}$$

The value of the Killing form on z is $\text{tr}(\text{ad}z)^2 = 2\lambda_i(z)^2 > 0$.

Therefore, the number k_e is equal to the number of summands of type \mathfrak{g}_{\diamond} + the number of summands of type $\mathfrak{so}(3, \mathbb{R})$ + the number of summands of type $\mathfrak{sl}(2, \mathbb{R})$ with a negative value of Killing form on the intersection $\mathfrak{sl}(2, \mathbb{R}) \cap \text{Ker } A$, while the number k_h is equal to the number of summands of type $\mathfrak{g}_{\diamond}^h$ + the number of summands of type $\mathfrak{sl}(2, \mathbb{R})$ with a positive value of Killing form on the intersection $\mathfrak{sl}(2, \mathbb{R}) \cap \text{Ker } A$, which proves the theorem.

References

- [1] F. Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.
- [2] I.M. Gel'fand and I.Ya. Dorfman. Hamiltonian operators and algebraic structures related to them. *Functional Analysis and Its Applications*, 13:248–262, 1979.
- [3] F. Magri and C. Morosi. *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*. University of Milano, 1984.

- [4] A.G. Reiman and M.A. Semenov-Tyan-Shanskii. A family of Hamiltonian structures, hierarchy of hamiltonians, and reduction for first-order matrix differential operators. *Functional Analysis and Its Applications*, 14:146–148, 1980.
- [5] A.V. Bolsinov. Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution. *Mathematics of the USSR-Izvestiya*, 38(1):69–90, 1992.
- [6] P.A. Damianou. Multiple Hamiltonian structures for Toda-type systems. *J. Math. Phys.*, 35(10):5511–5541, 1994.
- [7] Yu.B. Suris. On the bi-hamiltonian structure of Toda and relativistic Toda lattices. *Physics Letters A*, 180(6):419–429, 1993.
- [8] M.S. Alber, G.G. Luther, J.E. Marsden, and J.M. Robbins. Geometry and control of three-wave interactions. In *The Arnoldfest*, pages 55–80, 1997.
- [9] A.V. Bolsinov. Multidimensional Euler and Clebsch cases and Lie pencils. In *Tensor and Vector Analysis*, pages 25–30. Gordon and Breach Science Publ., Amsterdam, 1998.
- [10] I.D. Marshall. The Kowalevski top: its r-matrix interpretation and bihamiltonian formulation. *Communications in Mathematical Physics*, 191:723–734, 1998.
- [11] Yu.N. Fedorov. Integrable systems, Poisson pencils, and hyperelliptic Lax pairs. *Journal of Mathematical Sciences*, 94:1501–1511, 1999.
- [12] A.V. Bolsinov and A.V. Borisov. Compatible poisson brackets on Lie algebras. *Mathematical Notes*, 72:10–30, 2002.
- [13] A.T. Fomenko. Morse theory of integrable Hamiltonian systems. *Sov. Math., Dokl.*, 33:502–506, 1986.
- [14] A.T. Fomenko. The topology of surfaces of constant energy in integrable hamiltonian systems, and obstructions to integrability. *Mathematics of the USSR-Izvestiya*, 29(3):629, 1987.
- [15] A.V. Bolsinov and A.T. Fomenko. *Integrable Hamiltonian systems. Geometry, Topology and Classification*. CRC Press, 2004.
- [16] L.M. Lerman and Ya.L. Umanskij. Structure of the Poisson action of \mathbb{R}^2 on a four-dimensional symplectic manifold. I. *Selecta Math. Soviet.*, 6(4):365–396, 1987. Selected translations.
- [17] L.M. Lerman and Ya.L. Umanskij. The structure of the Poisson action of \mathbb{R}^2 on a four-dimensional symplectic manifold. II. *Selecta Math. Soviet.*, 7(1):39–48, 1988.
- [18] L.M. Lerman and Ya.L. Umanskij. Classification of four-dimensional integrable Hamiltonian systems and Poisson actions of \mathbb{R}^2 in extended neighbourhoods of simple singular points. I. *Russian academy of sciences, Sbornik mathematics*, 77(2):511–542, 1994.
- [19] L.M. Lerman and Ya.L. Umanskij. Classification of four-dimensional integrable Hamiltonian systems and Poisson actions of \mathbb{R}^2 in extended neighbourhoods of simple singular points. II. *Russian academy of sciences, Sbornik mathematics*, 78(2):479–506, 1994.
- [20] L.M. Lerman and Ya.L. Umanskij. Classification of four-dimensional integrable Hamiltonian systems and Poisson actions of \mathbb{R}^2 in extended neighbourhoods of simple singular points. III. Realization. *Sbornik: mathematics*, 186(10):1477–1491, 1995.

- [21] M.P. Kharlamov and T.I. Pogossian. Bifurcation set and integral manifolds of the problem concerning the motion of a rigid body in a linear force field. *Journal of Applied Mathematics and Mechanics*, 43(3):452 – 462, 1979.
- [22] C. Jacobi. *Vorlesungen über Dynamik*. Berlin, 1884.
- [23] J. Liouville. Note sur l'intégration des équations différentielles de la dynamique, présentée au bureau des longitudes le 29 juin 1853. *J. Math. Pures et Appl.*, 20:137–138, 1855.
- [24] H. Mineur. Réduction des systèmes mécaniques à n degrés de liberté admettant n intégrales premières uniformes en involution aux systèmes à variables séparées. *J. Math. Pure Appl.*, 15:221–267, 1936.
- [25] V. Arnold and A. Avez. *Ergodic Problems of Classical Mechanics*. Benjamin, Reading, Mass, 1967.
- [26] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer-Verlag, 1978.
- [27] P. Libermann and C.M. Marle. *Symplectic Geometry and Analytical Mechanics*. Kluwer Academic Publishers, 1987.
- [28] A.S. Mishchenko and A.T. Fomenko. Euler equations on finite-dimensional Lie groups. *Mathematics of the USSR-Izvestiya*, 12(2):371–389, 1978.
- [29] S.V. Manakov. Note on the integration of euler's equations of the dynamics of an n -dimensional rigid body. *Functional Analysis and Its Applications*, 10:328–329, 1976.
- [30] A.V. Bolsinov and A.A. Oshemkov. Bi-hamiltonian structures and singularities of integrable systems. *Regular and Chaotic Dynamics*, 14:431–454, 2009.
- [31] I. Basak. Bifurcation analysis of the Zhukovskii-Volterra system via bi-Hamiltonian approach. *Regular and Chaotic Dynamics*, 15:677–684, 2010.
- [32] J. Williamson. On the algebraic problem concerning the normal forms of linear dynamical systems. *American Journal of Mathematics*, 58(1):141–163, 1936.
- [33] L. Eliasson. *Normal forms for Hamiltonian systems with Poisson commuting integrals*. PhD thesis, University of Stockholm, 1984.
- [34] L. Eliasson. Normal forms for hamiltonian systems with Poisson commuting integrals - elliptic case. *Commentarii Mathematici Helvetici*, 65:4–35, 1990.
- [35] E. Miranda. *On symplectic linearization of singular Lagrangian foliations*. PhD thesis, Universitat de Barcelona, 2003.
- [36] R. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra and its Applications*, 147(0):323 – 371, 1991.
- [37] I. M. Gel'fand and I. S. Zakharevich. Spectral theory of a pencil of skew-symmetric differential operators of third order on S^1 . *Functional Analysis and Its Applications*, 23:85–93, 1989.
- [38] I. Kozlov. Elementary proof of Jordan-Kronecker theorem. *arXiv:1109.5371v1*, 2011.
- [39] I. Zakharevich. Kronecker webs, bihamiltonian structures, and the method of argument translation. *Transformation Groups*, 6:267–300, 2001.

- [40] A. Panasyuk. Bi-Hamiltonian structures with symmetries, Lie pencils and integrable systems. *J. Phys. A*, 42(16):165205, 20, 2009.
- [41] H. Cendra, J. Marsden, and T. Ratiu. Cocycles, compatibility, and poisson brackets for complex fluids. In *Advances in Multifield Theories for Continua with Substructure*, pages 51–73. Birkhauser Boston, 2004.
- [42] J.-P. Dufour and Nguyen Tien Zung. *Poisson Structures and Their Normal Forms (Progress in Mathematics)*. Birkhauser Basel, 2005.
- [43] A.A. Kirillov. *Lectures on the orbit method*. American Mathematical Society, Providence, 2004.